MULTI-SORTED LOGIC, MODELS AND LOGICAL GEOMETRY

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ABSTRACT. Let Θ be a variety of algebras, (H, Ψ, f) be a model, where H is an algebra from Θ, Ψ is a set of relation symbols φ, f is an interpretation of all φ in H. Let X^0 be an infinite set of variables, Γ be a collection of all finite subsets in X^0 (collection of sorts), $\widetilde{\Phi}$ be the multi-sorted algebra of formulas. These data define a knowledge base $KB(H, \Psi, f)$. In the paper the notion of isomorphism of knowledge bases is considered. We give sufficient conditions which provide isomorphism of knowledge bases. We also study the problem of necessary and sufficient conditions for isomorphism of two knowledge bases.

1. INTRODUCTION

Speaking about knowledge we proceed from its representation in three components.

(1) *Description of knowledge* presents a syntactical component of knowledge. From algebraic viewpoint description of knowledge is a set of formulas T in the algebra of formulas $\Phi(X)$, $X = \{x_1, \ldots, x_n\}$. Now we only note that $\Phi(X)$ is one of domains of multi-sorted algebra $\widetilde{\Phi}$ (detailed definition of $\widetilde{\Phi}$ see in [9], [12], [14] and Section 2.6).

(2) *Subject area of knowledge* is presented by a model (H, Ψ, f), where H is an algebra in fixed variety of algebras Θ, Ψ is a set of relation symbols φ and f is an interpretation of each φ in H.

(3) *Content of knowledge* is a subset in H^n, where H^n is the Cartesian power of H. Each content of knowledge A corresponds to the description of knowledge $T \subset \Phi(X)$, $|X| = n$. If we regard H^n as an

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affine space then this correspondence can be treated geometrically (see Section 2.3).

In order to describe the dynamic nature of a knowledge base we introduce two categories: the category of descriptions of knowledge $F_\Theta(f)$ and the category of knowledge contents $LG_\Theta(f)$. These categories are defined using the machinery of logical geometry (see Sections 2.4, 2.5, or [17]).

We shall emphasize that all of our notions are oriented towards an arbitrary variety of algebras Θ. Therefore, algebra, logic and geometry of knowledge bases are related to this variety. Universal algebraic geometry and logical geometry deal with algebras H from Θ, while logical geometry studies also arbitrary models (H, Ψ, f). Moreover, for each particular variety of algebras Θ there are its own interesting problems and solutions.

The objective of the present paper is to study connections between isomorphisms of knowledge bases and isotypeness of subject areas of knowledge.

Varying Θ, we arrive to numerous specific problems. In particular, if Θ is a variety of all quasigroups, it is interesting to understand the connection between logical isotypeness and isotopy of quasigroups [3], [18].

The paper consists of two parts. In the first one the necessary notions from logical geometry are introduced. In the second part, logical geometry is considered in the context of knowledge bases. In particular, we describe conditions on the models which provide an isomorphism of corresponding knowledge bases.

2. BASIC NOTIONS

2.1. **Points and affine spaces.** Let an algebra $H \in \Theta$ and a set $X = \{x_1, \ldots, x_n\}$ be given. A point $\overline{a} = (a_1, \ldots, a_n)$ can be represented as the map $\mu : X \to H$ such that $a_i = \mu(x_i)$. Denote by H^n the affine space consisting of such points.

Every map μ gives rise to the homomorphism $\mu : W(X) \to H$, where $W(X)$ is the free algebra over a set X in the variety Θ. Thus, every affine space can be considered as the set $Hom(W(X), H)$ of all homomorphisms from $W(X)$ to H.

Each point μ as a homomorphism has a kernel $Ker(\mu)$, which is a binary relation on the set $W(X)$. By definition, elements $w, w' \in W(X)$ belong to the binary relation $Ker(\mu)$ if and only if $w^\mu = (w')^\mu$, where w^μ is notation for $\mu(w)$.

We will also consider a logical kernel $LKer(\mu)$ of a point μ. A formula $u \in \Phi(X)$ belongs to $LKer(\mu)$, if the point μ satisfies the formula u.

2.2. Extended boolean algebras.

We start from the definition of an existential quantifier on a boolean algebra. Let B be a boolean algebra. Existential quantifier on B is a unary operation $\exists : B \to B$ such that the following conditions hold:

(1) $\exists\, 0 = 0$,
(2) $a \leq \exists a$,
(3) $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

Universal quantifier $\forall : B \to B$ is dual to $\exists : B \to B$, they are related by $\forall a = \neg(\exists(\neg a))$.

Definition 2.1. *Let a set of variables $X = \{x_1, \ldots, x_n\}$ and a set of relations Ψ be given. A boolean algebra B is called an extended boolean algebra over $W(X)$ if*

1. the existantial quantifier $\exists x$ is defined on B for all $x \in X$, and $\exists x \exists y = \exists y \exists x$ for all $x, y \in X$;

2. to every relation symbol $\varphi \in \Psi$ of arity n_φ and a collection of elements $w_1, \ldots, w_{n_\varphi}$ from $W(X)$ there corresponds a nullary operation (a constant) of the form $\varphi(w_1, \ldots, w_{n_\varphi})$ in B.

Thus, the signature L_X of extended boolean algebra consists of the boolean connectives, existential quantifiers $\exists x$ and of the set of constants $\varphi(w_1, \ldots, w_{n_\varphi})$:

$$L_X = \{\vee, \wedge, \neg, \exists x, M_X\},$$

where M_X is the set of all $\varphi(w_1, \ldots, w_{n_\varphi})$.

The algebra of formulas $\Phi(X)$ is the example of an extended boolean algebra (see [9], [12], [14]). A formula $w \equiv w'$ is one of the constants, where φ is the equality predicate "$\equiv$". Depending on the context, we call it equality or equation.

Consider another important example of extended boolean algebras. Let $(f) = (H, \Psi, f)$ be a model. Take the affine space $Hom(W(X), H)$ and denote by $Bool(W(X), H)$ the boolean algebra of all subsets of $Hom(W(X), H)$.

Let us define on this algebra the existential quantifier. If A is an element of $Bool(W(X), H)$ then the element $B = \exists x A$ is defined by the rule: a point μ belongs to B if there exists a point $\nu \in A$ such that $\mu(x') = \nu(x')$ for each $x' \in X$, $x' \neq x$.

Define now constants on $Bool(W(X), H)$. For a relational symbol φ of arity m denote by $[\varphi(w_1, \ldots, w_m)]_{(f)}$ the subset in $Bool(W(X), H)$ consisting of all points $\mu : W(X) \to H$ satisfying the relation $\varphi(w_1, \ldots, w_m)$.

This means that $(w_1^\mu, \ldots, w_m^\mu)$ belongs to the set $f(\varphi)$, where $f(\varphi)$ is a subset in H^m, consisting of all points which belong to φ under interpretation f.

Denote this extended boolean algebra by $Hal_\Theta^X(f)$. In particular, if Ψ consists solely of the equality predicate symbol, then the corresponding algebra is denoted by $Hal_\Theta^X(H)$.

In Section 2.6 we will define a homomorphism between $\Phi(X)$ and $Hal_\Theta^X(f)$:

$$Val_{(f)}^X : \Phi(X) \to Hal_\Theta^X(f),$$

with the property

$$Val_{(f)}^X(\varphi(w_1, \ldots, w_m)) = [\varphi(w_1, \ldots, w_m)]_{(f)}.$$

This homomorphism allows us to define algebraically such notions as "a point satisfies a formula" and "a logical kernel of a point". Such approach agrees with the model theoretic inductive one (see [6]).

Now we only observe, that for a formula $u \in \Phi(X)$ its image $Val_{(f)}^X(u)$ is defined as the set of points $\mu : W(X) \to H$ satisfying u. In this case, a formula $u \in \Phi(X)$ belongs to the logical kernel $LKer(\mu)$ of $\mu : W(X) \to H$ if and only if $\mu \in Val_{(f)}^X(u)$. Note also, that $LKer(\mu)$ is a boolean ultrafilter in the algebra of formulas $\Phi(X)$ containing X-elementary theory $Th^X(f)$ of the model (H, Ψ, f). In this sense, we say that $LKer(\mu)$ is an LG-type of the point μ (see [6] for the model theoretic definition of a type and [19] for LG-type). Recall that $Th^X(f)$ consists of all formulas $u \in \Phi(X)$ which hold true on each point $\mu : W(X) \to H$. Thus,

$$Th^X(f) = \bigcap_{\mu \in Hom(W(X), H)} LKer(\mu).$$

2.3. **Galois correspondence.** Define now a correspondence between sets T of formulas of the form $w \equiv w'$ in the algebra of formulas $\Phi(X)$ and subsets of points A from the affine space $Hom(W(X), H)$. We set $T'_H = A$, where A consists of all points $\mu : W(X) \to H$ such that $T \subset Ker(\mu)$. In other words, T'_H consists of all points satisfying all formulas from T. We call this T'_H *an algebraic set* defined by the set of formulas T.

On the other hand, for a given set of points A we define a set of formulas T as

$$T = A'_H = \bigcap_{\mu \in A} Ker(\mu).$$

By the definition, T is a congruence, it is called H-*closed congruence* defined by A. One can check that such correspondence between sets of

formulas of the form $w \equiv w'$ from the algebra $\Phi(X)$ and sets of points from the affine space $Hom(W(X), H)$ is the Galois correspondence [5].

Now we consider the case of arbitrary set of formulas $T \subset \Phi(X)$. Let $T^L_{(f)} = A$ be a set of all points $\mu : W(X) \to H$ such that $T \subset LKer(\mu)$. The set $T^L_{(f)}$ is called *a definable set* presented by the set of formulas T. Let now A be a set of points from $Hom(W(X), H)$. We define $A^L_{(f)}$ as

$$A^L_{(f)} = T = \bigcap_{\mu \in A} LKer(\mu).$$

Direct calculations show that $u \in A^L_{(f)}$ if and only if $A \subset Val^X_{(f)}(u)$. Note that $A^L_{(f)}$ is a filter in $\Phi(X)$ called H-*closed filter* defined by the set A.

Thus, the Galois correspondences described above give rise to universal algebraic geometry if T is a set of equalities, and to logical geometry if T is an arbitrary set of formulas.

Recall that a set A from $Hom(W(X), H)$ is Galois-closed if $A''_H = A$ or $A^{LL}_{(f)} = A$, depending on the given Galois correspondence. A congruence T on $W(X)$ is Galois-closed if $T''_H = T$, a filter T in $\Phi(X)$ is Galois-closed if $T^{LL}_{(f)} = T$.

So, we have a one-to-one correspondence between algebraic sets in $Hom(W(X), H)$ and closed congruences on $W(X)$, between definable sets in $Hom(W(X), H)$ and closed filters in the extended boolean algebra $\Phi(X)$.

2.4. **Some categories.** In this section we define various categories, which are necessary for further considerations.

2.4.1. *Categories* Θ^0, $\widetilde{\Phi}$ *and* $\Theta^*(H)$. Let an infinite set of variables X^0 and a collection Γ of finite subsets of X^0 be given.

Denote by Θ^0 the category of all free algebras $W(X)$ in Θ, $X \in \Gamma$. Morphisms in this category are homomorphisms $s : W(X) \to W(Y)$.

Along with free algebras $W(X)$ we consider algebras of formulas $\Phi(X)$, which are also associated with the variety Θ. We define a category $\widetilde{\Phi}$ of all $\Phi(X)$, $X \in \Gamma$ in such a way that to each morphism $s : W(X) \to W(Y)$ it corresponds a morphism $s_* : \Phi(X) \to \Phi(Y)$ and this correspondence gives rise to a covariant functor from Θ^0 to $\widetilde{\Phi}$.

Define now the category $\Theta^*(H)$ of affine spaces over $H \in \Theta$. Objects of this category are affine spaces $Hom(W(X), H)$, morphisms are maps:

$$\widetilde{s} : Hom(W(X), H) \to Hom(W(Y), H),$$

where

$$s : W(Y) \to W(X)$$

are morphisms in the category of free algebras Θ^0.

For a point $\mu : W(X) \to H$ the point $\nu = \widetilde{s}(\mu) : W(Y) \to H$ is defined as follows:

$$\widetilde{s}(\mu) = \mu s : W(Y) \to H,$$

that is, $\nu(w) = \mu(s(w))$, $w \in W(Y)$.

Passages $W(X) \to Hom(W(X), H)$ and $s \to \widetilde{s}$ give rise to a contravariant functor

$$\Theta^0 \to \Theta^*(H).$$

There is the following

Theorem 2.2 ([7]). *The functor $\Theta \to \Theta^*(H)$ defines a duality of categories if and only if the algebra H generates the variety of algebras Θ, i.e., $\Theta = Var(H)$.*

2.4.2. *Categories $Hal_\Theta(H)$ and $Hal_\Theta(f)$.* For a given model (H, Ψ, f) we define categories $Hal_\Theta(H)$ and $Hal_\Theta(f)$. The first category is related to universal algebraic geometry, while the second one to logical geometry.

Objects of these categories are algebras $Hal_\Theta^X(H)$ and $Hal_\Theta^X(f)$, respectively. The categories $Hal_\Theta(H)$ and $Hal_\Theta(f)$ have different objects, since the sets of constants in algebras $Hal_\Theta^X(H)$ and $Hal_\Theta^X(f)$ are different (see Section 2.2).

Denote morphisms for both categories by s_*. A homomorphism $s : W(Y) \to W(X)$ gives rise to a map

$$\widetilde{s} : Hom(W(X), H) \to Hom(W(Y), H).$$

In its turn, $\widetilde{s}$ defines a morphism

$$s_* : Bool(W(Y), H) \to Bool(W(X), H)$$

by the rule: for an arbitrary $B \subset Hom(W(Y), H)$ we put

$$s_* B = \widetilde{s}^{-1}(B) = A \subset Hom(W(X), H).$$

Thus, A is a full pre-image of B under $\widetilde{s}$, it consists of all points μ from $Hom(W(X), H)$ such that $\widetilde{s}(\mu) = \mu s \in B$.

We would like to link together categories $\widetilde{\Phi}$ and $Hal_\Theta(f)$. Let $s : W(Y) \to W(X)$, $s_* : \Phi(Y) \to \Phi(X)$ and $v \in \Phi(Y)$ be given.

Proposition 2.3. *A point $\mu : W(X) \to H$ satisfies the formula $u = s_* v$ if and only if μs satisfies the formula v.*

Proof. In fact, this result follows from axiom (5) in Definition 2.9, which regulates the action of morphism s_* on formulas of the form $\varphi(w_1, \ldots, w_m)$. These formulas generate freely the algebra $\widetilde{\Phi}$ as a multi-sorted algebra (see Section 2.6 or [9]). $\square$

Let A be the set of all points satisfying the formula $u = s_* v \in \Phi(X)$, B be the set of all points satisfying the formula $v \in \Phi(Y)$.

Proposition 2.4. *Let $A_0 = s_* B = \widetilde{s}^{-1}(B)$. Then $A_0 = A$.*

Proof. Let $\mu' \in Hom(W(X), H)$ belongs to $\widetilde{s}^{-1}(B) = A_0$. By the definition this means that $\widetilde{s}(\mu') = \mu' s \in B$. Thus, $\mu' s$ satisfies the formula v. By Proposition 2.3, the point μ' satisfies the formula u. Hence, $\mu' \in A$. $\square$

We call a set A *s-closed* if $A_0 = A$, that is, $s_*(\widetilde{s}A) = A$. As follows from Proposition 2.4, each definable set is s-closed.

Consequently, we have the commutative diagram

$$
(1) \qquad
\begin{array}{ccc}
\Phi(Y) & \xrightarrow{\ \ s_*\ \ } & \Phi(X) \\
{\scriptstyle Val^Y_{(f)}}\Big\downarrow & & \Big\downarrow{\scriptstyle Val^X_{(f)}} \\
Hal_\Theta(f_2) & \xrightarrow{\ s_* = \widetilde{s}^{-1}\ } & Hal_\Theta(f_1),
\end{array}
$$

The commutativity of this diagram means that if $v \in \Phi(Y)$, $u = s_* v \in \Phi(X)$, $A = Val^X_{(f)}(u)$, $B = Val^Y_{(f)}(v)$, then $Val^X_{(f)}(s_* v) = s_* Val^Y_{(f)}(v)$.

From the categorical viewpoint, commutative diagram (1) determines a covariant functor from $\widetilde{\Phi}$ to $Hal_\Theta(f)$. From the point of view of multi-sorted algebras, the last equality means that $Val_{(f)}$ is a homomorphism of multi-sorted Halmos algebras.

2.4.3. *Categories $AG_\Theta(H)$ and $LG_\Theta(f)$.* The first category is related to algebraic sets in universal algebraic geometry, while the second one to definable sets in logical geometry.

Objects of the category $AG_\Theta(H)$ are partially ordered sets $AG^X_\Theta(H)$ of all algebraic sets in $Hom(W(X), H)$ with fixed X.

For a given homomorphism $s : W(Y) \to W(X)$, a morphism

$$\widetilde{s}_* : AG^X_\Theta(H) \to AG^Y_\Theta(H)$$

is defined as follows. Let A be an algebraic set in $Hom(W(X), H)$. Then $\widetilde{s}_* A = B$ is an algebraic set determined by the set of points of the form $\nu = \mu s$, where $\mu \in A$. In other words, B is the Galois closure of this set of points, i.e., $B = \widetilde{s}_* A = (\widetilde{s}A)''_H$. Morphisms, defined in such a way, preserve the partial order relation.

Objects of $LG_\Theta(f)$ are sets of all definable sets in $Hom(W(X), H)$ with fixed X. We assume, that each object $LG^X_\Theta(f)$ is a lattice.

Define morphisms in $LG_\Theta(f)$ as:

$$\widetilde{s}_* : LG^X_\Theta(H) \to LG^Y_\Theta(H).$$

Let A be a definable set in $Hom(W(X), H)$. Then $\widetilde{s}_* A = B$ is a definable set given by the set of points of the form $\nu = \mu s$, where $\mu \in A$. In other words, B is the Galois closure of this set of points, i.e., $B = \widetilde{s}_* A = (\widetilde{s} A)_{(f)}^{LL}$.

2.4.4. *Categories* $C_\Theta(H)$ *and* $F_\Theta(f)$. Objects of $C_\Theta(H)$ are partially ordered sets of H-closed congruences on $W(X)$. They are in one-to-one correspondence with the objects $AG_\Theta^X(H)$. Morphism in $C_\Theta(H)$

$$\widehat{s}_* : C_\Theta^Y(H) \to C_\Theta^X(H)$$

is defined using the maps between H-closed congruences in $C_\Theta^Y(H)$ and $C_\Theta^X(H)$. Let T_2 be an H-closed congruence in $C_\Theta^Y(H)$. Specify T_1 as an H-closed congruence in $C_\Theta^X(H)$ defined by the set of all equations of the form $s_*(w \equiv w')$, for all $w \equiv w'$ from T_2. In other words, $T_1 = (s_* T_2)_H''$.

Objects of the category $F_\Theta(f)$ are lattices of H-closed filters. We define morphisms in $F_\Theta(H)$

$$\widehat{s}_* : F_\Theta^Y(H) \to F_\Theta^X(H),$$

using the maps between H-closed filters in $F_\Theta^Y(H)$ and $F_\Theta^X(H)$. Let T_2 be an H-closed filter in $F_\Theta^Y(H)$. Determine T_1 as the H-closed filter in $F_\Theta^Y(H)$ defined by the set of formulas of the form $s_* v$, for all v from T_2, that is, $T_1 = (s_* T_2)_{(f)}^{LL}$.

2.5. **Relation between categories** $LG_\Theta(f)$ **and** $F_\Theta(f)$. We would like to determine the duality of categories $LG_\Theta(f)$ and $F_\Theta(f)$. According to their Galois correspondence there is a one-to-one correspondence between objects of these categories.

Let a homomorphism $s : W(Y) \to W(X)$ and a definable set B_0 from $Hom(W(Y), H)$ be given.

Define the set A_0 as the full pre-image of B_0 under $\widetilde{s}$, i.e., $A_0 = \widetilde{s}^{-1}(B_0)$ (see Section 2.4.2). Let B be a definable set such that $B = \widetilde{s}_* A_0 = (\widetilde{s} A_0)^{LL}$. Since $\widetilde{s} A_0 \subset B$, then $B = (\widetilde{s} A_0)^{LL} \subset B_0^{LL} = B_0$.

Define the H-closed filter T_2 as $T_2 = B^L$. Then, s_* and T_2 determine the H-closed filter $T_1 = (s_* T_2)^{LL} = \widehat{s}_* T_1$. Finally, we put $A = T_1^L$.

There is the commutative diagram:

$$
\begin{array}{ccc}
T_2 & \xrightarrow{\ \widehat{s}_*\ } & T_1 \\
{\scriptstyle Val_{(f)}^Y}\Big\downarrow & & \Big\downarrow{\scriptstyle Val_{(f)}^X} \\
B & \xleftarrow{\ \widetilde{s}_*\ } & A
\end{array}
$$

(2)

Indeed, since objects A_0, B, T_2, T_1 are defined uniquely by B_0 and $s : W(Y) \to W(X)$, for the commutativity of the diagram it is enough to check that $A_0 = A$. But this equality follows from Proposition 2.4.

Moreover, $\mu \in A_0$ if and only if $\mu s \in B$. In its turn, $\mu s \in B$ if and only if μs satisfies each formula $v \in T_2$. By Proposition 2.3, μs satisfies $v \in T_2$ if and only if μ satisfies $u = s_* v \in T_1$. Since A consists of all points satisfying all formulas $u = s_* v \in T_1$ then $A_0 = A$.

From diagram (2) follows that for each formula $v \in T_2$ there is the relation

$$Val_{(f)}^Y = \widetilde{s}_* Val_{(f)}^X s_*.$$

Definition 2.5. *A map $\alpha : A \to B$ of definable sets is called generalized regular if there is a map $\widetilde{s}_* : A \to B$ satisfying commutative diagram (2) such that $\alpha(\mu) = \widetilde{s}_*(\mu)$, for all $\mu \in A$.*

By the definition of the map $\widetilde{s}_*$, the image of a definable set under generalized regular map is a definable set. Thus, $LG_\Theta(f)$ is the category of lattices of definable sets with generalized regular maps as morphisms.

The similar approach works for the category of algebraic sets $AG_\Theta(H)$. So, we have a particular case of diagram (2):

$$
\begin{array}{ccc}
T_2 & \xrightarrow{\ s_*\ } & T_1 \\
{\scriptstyle Val_H^Y} \downarrow & & \downarrow {\scriptstyle Val_H^X} \\
B & \xleftarrow{\ \widetilde{s}_*\ } & A,
\end{array}
$$

where T_1 and T_2 are the Galois-closed congruences.

Definition 2.6. *A map $\alpha : A \to B$ of algebraic sets is called regular if there is a map $\widetilde{s}_* : A \to B$ satisfying the commutative diagram above such that $\alpha(\mu) = \widetilde{s}_*(\mu)$, for all $\mu \in A$.*

Thus, $AG_\Theta(H)$ is the category of partially ordered algebraic sets with regular maps as morphisms.

Summarizing, we have the theorem.

Theorem 2.7. *Let $Var(H) = \Theta$. The category $F_\Theta(f)$ of lattices of H-closed filters is anti-isomorphic to the category $LG_\Theta(f)$ of lattices of definable sets. The category $C_\Theta(H)$ of partially ordered congruences is anti-isomorphic to the category $AG_\Theta(H)$ of partially ordered algebraic sets.*

Proof. The proof of Theorem 2.7 follows from diagram (2). The condition $Var(H) = \Theta$ ensures that the homomorphism $s : W(Y) \to W(X)$ uniquely defines morphism $\widetilde{s}_*$.

$\square$

2.6. **Multi-sorted Halmos algebras.** In Section 2.4.2 we defined the categories $Hal_\Theta(H)$ and $Hal_\Theta(f)$. There is a natural way to treat these categories as multi-sorted algebras (see [8], [14]). We put

$$Hal_\Theta(H) = (Hal_\Theta^X(H), X \in \Gamma),$$

$$Hal_\Theta(f) = (Hal_\Theta^X(f), X \in \Gamma).$$

In this case, objects of the categories are presented as domains of multi-sorted algebras, while morphisms s_* are unary operations between domains. These algebras are Halmos algebras.

Remark 2.8. *We widely use the name P. Halmos, because he was one of the creators of algebraic logic. He introduced the important notion of a polyadic algebra. Along with other notions of universal algebra and universal algebraic geometry, the notion of a polyadic algebra gave rise to the theory, which, in particular, is used in this paper.*

For the precise definition of a multi-sorted Halmos algebra, first of all, we specify a signature of such algebras.

Let a finite set X from Γ, a variety Θ, an algebra $H \in \Theta$ and a set of relation symbols Ψ be given. The signature L^Ψ of a multi-sorted Halmos algebra $\mathfrak{L} = (\mathfrak{L}_X, X \in \Gamma)$ includes the signature of extended boolean algebras L_X (see Section 2.2) and operations of the form $s_* : \mathfrak{L}_X \to \mathfrak{L}_Y$, which correspond to morphisms $s : W(Y) \to W(X)$ in Θ^0.

Definition 2.9. *A multi-sorted algebra $\mathfrak{L} = (\mathfrak{L}_X, X \in \Gamma)$ in the signature L^Ψ is a Halmos algebra if*

(1) *Each domain $\mathfrak{L}_X$ is an extended boolean algebra in the signature L_X.*

(2) *Each map $s_* : \mathfrak{L}_X \to \mathfrak{L}_Y$ is a homomorphism of boolean algebras.*

(3) *For given $s_{1*} : \mathfrak{L}_X \to \mathfrak{L}_Y$ and $s_{2*} : \mathfrak{L}_Y \to \mathfrak{L}_Z$ there is the equality:*

$$s_{1*}s_{2*} = (s_1 s_2)_*.$$

In other words, it means that the correspondence $W(X) \to \mathfrak{L}_X$ and $s \to s_*$ define a covariant functor from the category Θ^0 to the category $\mathfrak{L}$.

(4) *Next two axioms control the interaction of s_* with quantifiers:*

(a) *Let $s_1 : W(X) \to W(Z)$ and $s_2 : W(X) \to W(Z)$ be given. Suppose, that $s_1(y) = s_2(y)$ for all $y \neq x$, $x, y \in X$. Then*

$$s_{1*}\exists x a = s_{2*}\exists x a, \ a \in \mathfrak{L}_X.$$

(b) *Let $s : W(X) \to W(Y)$ and $s(x) = y$ be given, $x \in X$, $y \in Y$. Let $x' \neq x$, $x' \in X$. Suppose, that $s(x') = w$, where $w \in W(Y)$, and y does not belong to the support of w. This condition means, that y does not participate in the shortest expression of the element $s(x') \in W(Y)$. Then*

$$s_*(\exists x a) = \exists(s(x))(s_* a), \ a \in \mathfrak{L}_X.$$

(5) *Let a relation symbol $\varphi \in \Psi$ of arity m and $s : W(X) \to W(Y)$ be given. Then*

$$s_*(\varphi(w_1, \ldots, w_m)) = \varphi(s w_1, \ldots, s w_m).$$

In particular, for each equation $w \equiv w'$ we have

$$s_*(w \equiv w') = (s(w) \equiv s(w')).$$

Halmos algebras constitute a variety, denote it by Hal_Θ. Moreover, the following fact takes place.

Theorem 2.10. *Let a model $(f) = (H, \Psi, f)$, $H \in \Theta$, be given. The variety Hal_Θ is generated by all algebras $Hal_\Theta(f)$ for all $H \in \Theta$.*

Now we give a more precise definition of the algebra $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$ and homomorphism $Val_{(f)}$. For the detailed constructions of $\widetilde{\Phi}$ and $Val_{(f)}$ see [9], [12], [13], [14].

Let φ denote a relation symbol of arity m from Ψ, M_X be the set of all $\varphi(w_1, \ldots, w_m)$, $w_i \in W(X)$.

The algebra $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$ is the free algebra generated by multi-sorted set $M = (M_X, X \in \Gamma)$ in the variety Hal_Θ.

For each X we define a map

$$M_X \to Hal_\Theta^X(f)$$

by the rule

$$\varphi(w_1, \ldots, w_m) \to [\varphi(w_1, \ldots, w_m)]_{(f)}.$$

It induces the map of multi-sorted sets

$$M \to Hal_\Theta(f).$$

Since M generates freely the algebra $\widetilde{\Phi}$, then the last map can be extended up to the homomorphism of multi-sorted algebras

$$Val_{(f)} : \widetilde{\Phi} \to Hal_\Theta(f).$$

On components we have

$$Val_{(f)}^X : \Phi(X) \to Hal_\Theta^X(f).$$

Note that the algebra $\widetilde{\Phi}$ can be defined semantically. Let $\mathfrak{L}^0$ be an absolutely free algebra generated by the set M in the signature L^Ψ.

Let a model $(f) = (H, \Psi, f)$ and the corresponding algebra $Hal_\Theta(f)$ be given. We will treat the algebras $Hal_\Theta(f)$ as universal realization of the algebra $\mathfrak{L}^0$.

With each element $\varphi(w_1, \ldots, w_m) \in \mathfrak{L}^0$ we associate an element $[\varphi(w_1, \ldots, w_m)]_{(f)} \in Hal_\Theta(f)$. This correspondence gives rise to the homomorphism

$$Val^0_{(f)} : \mathfrak{L}^0 \to Hal_\Theta(f).$$

Denote by $\rho_{(f)}$ the kernel of this homomorphism. Note that it coincides with the set of identities of the algebra $Hal_\Theta(f)$. Let us consider the congruence

$$\rho = \bigcap_{(f)} \rho_{(f)}.$$

Since $Val^0_{(f)}$ is a unique homomorphism from $\mathfrak{L}^0$ to $Hal_\Theta(f)$ and all algebras of the form $Hal_\Theta(f)$ generate the variety Hal_Θ, then

$$\widetilde{\Phi} = \mathfrak{L}^0/\rho.$$

This expression gives rise to a description of the algebra $\widetilde{\Phi}$ which allows us to calculate images of the elements from $\mathfrak{L}^0$ in the algebra $Hal_\Theta(f)$. In this sense, this is a semantical definition of $\widetilde{\Phi}$.

All above can be summarized in the diagram

$$\begin{array}{ccc} \mathfrak{L}^0 & \xrightarrow{\ Val^0_{(f)}\ } & Hal_\Theta(f) \\ & {\scriptstyle\rho}\searrow \quad \nearrow{\scriptstyle Val_{(f)}} & \\ & \widetilde{\Phi} & \end{array}$$

3. LOGICAL GEOMETRY AND KNOWLEDGE BASES

3.1. **From logic and geometry to knowledge theory.** In the previous section we introduced a necessary system of notions. All these concepts naturally arise and interact in a certain order. The further exposition will be related to applications to knowledge bases.

3.2. **Knowledge bases.** From now on we will treat categories $F_\Theta(f)$ and $LG_\Theta(f)$ as *the categories of description of a knowledge and content of a knowledge*, accordingly.

Recall that we distinguished three components of knowledge representation:

- *description of knowledge,*
- *subject area of knowledge,*
- *content of knowledge.*

The next three mathematical objects correspond to these components:

- *the category of lattices of H-closed filters $F_\Theta(f)$,*
- *a model (H, Ψ, f),*
- *the category of lattices of definable sets $LG_\Theta(f)$.*

Definition 3.1. *A knowledge base $KB = KB(H, \Psi, f)$ is a triple $(F_\Theta(f), LG_\Theta(f), Ct_f)$, where $F_\Theta(f)$ is the category of description of knowledge, $LG_\Theta(f)$ is the category of content of knowledge, and*

$$Ct_f : F_\Theta(f) \to LG_\Theta(f)$$

is a contravariant functor.

The functor Ct_f transforms the knowledge description to the knowledge content. Morphisms of the categories $F_\Theta(f)$ and $LG_\Theta(f)$ make knowledge bases a dynamical object.

Remark 3.2. *We use the term "knowledge bases" instead of a more precise "a knowledge base model".*

For a given model $(f) = (H, \Psi, f)$, each concrete knowledge is a triple (X, T, A), where $X \in \Gamma$, T is a set of formulas from $\Phi(X)$ and A is the set of points from $Hom(W(X), H)$ such that $A = T_{(f)}^L = (T_{(f)}^{LL})^L$. Therefore, T and $T_{(f)}^{LL}$ describe the same content A.

3.3. **Isomorphism of knowledge bases.** The definition of an isomorphism of two knowledge bases KB_1 and KB_2 assumes an isomorphism of categories of knowledge content, which implies the isomorphism of categories of descriptions of knowledge $F_\Theta(f_1)$ and $F_\Theta(f_2)$. Thus,

Definition 3.3. *Knowledge bases $KB_1 = KB(H_1, \Psi, f_1)$ and $KB_2 = KB(H_2, \Psi, f_2)$ are called isomorphic if they match the commutative diagram*

$$
\begin{array}{ccc}
F_\Theta(f_1) & \xrightarrow{\ \alpha\ } & F_\Theta(f_2) \\
{\scriptstyle Ct_{f_1}}\downarrow & & \downarrow{\scriptstyle Ct_{f_2}} \\
LG_\Theta(f_1) & \xrightarrow{\ \beta\ } & LG_\Theta(f_2),
\end{array}
$$

where α and β are isomorphisms of categories.

Let us return to the ideas of logical geometry with respect to knowledge bases. We will use some material from [1].

Definition 3.4. *Models $(f_1) = (H_1, \Psi, f_1)$ and $(f_2) = (H_2, \Psi, f_2)$ are called LG-equivalent, if for each $X \in \Gamma$ and $T \subset \Phi(X)$ the following equality takes place*

$$T_{(f_1)}^{LL} = T_{(f_2)}^{LL}.$$

Recall that the logical kernel $LKer(\mu)$ of a point $\mu \in Hom(W(X), H)$ is X-LG-type of μ. Denote by $S^X(f)$ the set of all X-LG-types of the model (f).

Definition 3.5. *Models* $(f_1) = (H_1, \Psi, f_1)$ *and* $(f_2) = (H_2, \Psi, f_2)$ *are called LG-isotypic, if*

$$S^X(f_1) = S^X(f_2),$$

for each finite $X \in \Gamma$.

In other words, models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) are LG-isotypic, if the subject areas of algebras H_1 and H_2 have equal possibilities with respect to solution of logical formulas from $T \subset \Phi(X)$ for each finite $X \in \Gamma$. The notions of LG-isotypeness and LG-equivalence are tightly connected.

Theorem 3.6 ([1]). *Models* (H_1, Ψ, f_1) *and* (H_2, Ψ, f_2) *are LG-equivalent if and only if they are LG-isotypic.*

Remark 3.7. *Isotypiness of models imposes some constraints on interpretations* f_1 *and* f_2. *Let a point* $\mu \in Hom(W(X), H_1)$ *satisfies a formula* $u = \varphi(w_1, \ldots, w_m)$, $\varphi \in \Psi$. *Then* $\nu \in Hom(W(X), H_2)$ *satisfies the same* u. *Thus* $(w_1^\mu, \ldots, w_m^\mu)) \in f_1(\varphi) \subset H_1^m$ *if and only if* $(w_1^\nu, \ldots, w_m^\nu) \in f_2(\varphi) \subset H_2^m$. *In particular,* $w_i^\mu = w_j^\mu$ *if and only if* $w_i^\nu = w_j^\nu$.

The next theorem ties together isotypeness of models and isomorphism of knowledge bases.

Theorem 3.8. *If models* (H_1, Ψ, f_1) *and* (H_2, Ψ, f_2) *are isotypic then the corresponding knowledge bases are isomorphic.*

Proof. By Theorem 3.6, isotypic models are LG-equaivalent. Theorem 6.12 from [1] states that if the models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) are LG-equivalent then the categories $LG_\Theta(f_1)$ and $LG_\Theta(f_2)$ are isomorphic. From diagram (2) follows that the categories $F_\Theta(f_1)$ and $F_\Theta(f_2)$ are isomorphic. In fact, they coincide and isomorphism α is the identity isomorphism of the categories. Therefore, knowledge bases $KB_1 = KB(H_1, \Psi, f_1)$ and $KB_2 = KB(H_2, \Psi, f_2)$ are isomorphic. $\square$

Definition 3.9. *Knowledge bases* $KB_1 = KB(H_1, \Psi, f_1)$ *and* $KB_2 = KB(H_2, \Psi, f_2)$ *are called isotypic if the models* (H_1, Ψ, f_1) *and* (H_2, Ψ, f_2) *are isotypic.*

According to Theorem 3.8, isotypic knowledge bases are isomorphic. Theorem 3.8 generalizes the theorem from [16], which states that knowledge bases over finite automorphic models are informationally

equivalent. It also generalizes the result from [1] about informational equivalence of isotypic knowledge bases.

Let us treat the isomorphism problem for knowledge bases from a slightly different angle.

Definition 3.10. *Let φ_1 and φ_2 be functors from a category C_1 to a category C_2. We will say that an isomorphism of functors $S : \varphi_1 \to \varphi_2$ is given, if for each morphism $\nu : A \to B$ from C_1 the following commutative diagram takes place*

$$\begin{array}{ccc}
\varphi_1(A) & \xrightarrow{\ S_A\ } & \varphi_2(A) \\
{\scriptstyle\varphi_1(\nu)}\downarrow & & \downarrow{\scriptstyle\varphi_2(\nu)} \\
\varphi_1(B) & \xrightarrow{\ S_B\ } & \varphi_2(B).
\end{array}$$

Here S_A is A-component of S, i.e., S_A is a function which provides a bijection between $\varphi_1(A)$ and $\varphi_2(A)$. The same condition holds true for S_B.

An invertible functor from a category to itself is called *an automorphism* of a category. An automorphism φ of a category C is called *inner* (see [10]) if φ is isomorphic to the identity functor 1_C.

For each model (H, Ψ, f) the correspondence $s \to s_*$ gives rise to functors

$$Cl_H : \Theta^0 \to PoSet,$$
$$Cl_{(f)} : \widetilde{\Phi} \to Lat,$$

where $PoSet$ is the category of partially ordered sets, Lat is the category of lattices. The functor Cl_H assigns a partially ordered set $C_\Theta^X(H)$ of all H-closed congruences on $W(X)$ to each $W(X)$, while $Cl_{(f)}$ assigns a lattice of H-closed filters in $\Phi(X)$ to each $\Phi(X)$.

Let us consider the commutative diagram

$$\begin{array}{ccc}
\Theta^0 & \xrightarrow{\ \varphi\ } & \Theta^0 \\
& {\scriptstyle Cl_{H_1}}\searrow \quad \swarrow{\scriptstyle Cl_{H_2}} & \\
& PoSet &
\end{array}$$

where φ is an automorphism of the category Θ^0. Commutativity of this diagram means that there is an isomorphism of functors

$$\alpha_\varphi : Cl_{H_1} \to Cl_{H_2} \cdot \varphi.$$

This isomorphism of functors means that the following diagram is commutative

$$\begin{array}{ccc}
Cl_{H_1}(W(Y)) & \xrightarrow{\ (\alpha_\varphi)_{W(Y)}\ } & Cl_{H_2}(\varphi(W(Y)) \\
{\scriptstyle Cl_{H_1}(s)}\downarrow & & \downarrow{\scriptstyle Cl_{H_2}(\varphi(s))} \\
Cl_{H_1}(W(X)) & \xrightarrow{\ (\alpha_\varphi)_{W(X)}\ } & Cl_{H_2}(\varphi(W(X)).
\end{array}$$

Similarly, the commutative diagram

$$\widetilde{\Phi} \xrightarrow{\quad\varphi\quad} \widetilde{\Phi}$$

$$Cl_{(f_1)} \searrow \qquad \swarrow Cl_{(f_2)}$$

$$Lat$$

gives rise to the isomorphism of functors

$$\alpha_\varphi : Cl_{(f_1)} \to Cl_{(f_2)}\varphi.$$

Definition 3.11 ([15])**.** *Algebras H_1 and H_2 from a variety Θ are called geometrically automorphically equivalent if for some automorphism φ of the category Θ^0 there is the functor isomorphism $\alpha_\varphi : Cl_{H_1} \to Cl_{H_2} \cdot \varphi$.*

Definition 3.12 ([15])**.** *Models (H_1, Ψ, f_1) and (H_2, Ψ, f_2), where $H_1, H_2 \in \Theta$, are called logically automorphically equivalent if for some automorphism φ of the category $\widetilde{\Phi}$ there is the functor isomorphism $\alpha_\varphi : Cl_{(f_1)} \to Cl_{(f_2)} \cdot \varphi$.*

In the case of geometry over algebras, the following theorem is valid.

Theorem 3.13 ([15])**.** *Let $Var(H_1) = Var(H_2) = \Theta$. If algebras H_1 and H_2 are geometrically automorphically equivalent then the categories $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$ are isomorphic.*

A generalization of this result for the case of logical geometry and models is of great interest. Here is the corresponding result (for the proof see [2]).

Theorem 3.14. *Let (H_1, Ψ, f_1) and (H_2, Ψ, f_2) be logically automorphically equivalent models such that $Var(H_1) = Var(H_2) = \Theta$. Then the categories $LG_\Theta(f_1)$ and $LG_\Theta(f_2)$, and the corresponding knowledge bases $KB_1 = KB(H_1, \Psi, f_1)$ and $KB_2 = KB(H_2, \Psi, f_2)$ are isomorphic.*

The following problem arises in a natural way.

Problem 3.15. *Find necessary and sufficient conditions on models (H_1, Ψ, f_1) and (H_2, Ψ, f_2), which provide an isomorphism of the corresponding knowledge bases.*

Theorem 3.8 gives a sufficient condition for knowledge bases isomorphism.

The following proposition plays an important role.

Proposition 3.16 ([11])**.** *Assume that for a variety Θ each automorphism of the category Θ^0 is inner. The categories of algebraic sets $AG_\Theta(H_1)$ and $AG_\Theta(H_2)$, where H_1, $H_2 \in \Theta$, are isomorphic if and only if the algebras H_1, H_2 are AG-equivalent.*

For knowledge bases Proposition 3.16 gives necessary and sufficient conditions for knowledge bases isomorphism when the set of relation symbols Ψ of the corresponding knowledge base $KB(H, \Psi, f)$ contains only equality predicate symbol. The general case is still open problem.

Problem 3.17. *Let models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) be given, $H_1, H_2 \in \Theta$. Assume that for the variety Θ each automorphism of the category $\widetilde{\Phi}$ is inner. Is it true that $LG_\Theta(f_1)$ and $LG_\Theta(f_2)$ are isomorphic if and only if H_1 and H_2 are LG-equivalent?*

Of course, the necessary and sufficient conditions depend on the variety Θ. In this respect it is interesting to consider the following problem.

Problem 3.18. *What are automorphisms of the category $\widetilde{\Phi}$ for various varieties Θ.*

Note that for applications the varieties of groups and semigroups are of special interest. We finish our discussion with the question, which is also important for applications.

Problem 3.19. *What are necessary and sufficient conditions providing an isomorphism of finite knowledge bases.*

References

[1] E. Aladova, E.Plotkin, T. Plotkin, *Isotypeness of models and knowledge bases equivalence*, Mathematics in Computer Science, 7(4) (2013), 421–438.

[2] E. Aladova, B.Plotkin, T. Plotkin, *Similarity of models and knowledge bases isomorphism*, Preprint.

[3] A. A. Gvaramiya, *Halmos algebras and axiomatizable classes of quasigroups* (Russian) Uspekhi Mat. Nauk **40** (1985), no. 4(244), 215–216.

[4] P.R. Halmos, *Algebraic logic*, New York, (1969).

[5] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, **5**, Springer-Verlag, New York-Berlin, (1971).

[6] D. Marker, *Model Theory: An Introduction*, Springer Verlag, (2002).

[7] G. Mashevitzky, B. Plotkin, E. Plotkin, Automorphisms of categories of free algebras of varieties, *Electronic Research Announcements of AMS*, **8** (2002) 1–10.

[8] B. Plotkin, *Universal algebra, algebraic logic and databases.* Kluwer Acad. Publ., 1994.

[9] B.Plotkin, Seven lectures on the universal algebraic geometry, Preprint,(2002), Arxiv:math, GM/0204245, 87pp.

[10] B. Plotkin, *Algebras with the same algebraic geometry*, Proceedings of the Steklov Institute of Mathematics, MIAN, **242** (2003), 176–207.

[11] B. Plotkin. Varieties of algebras and algebraic varieties. Categories of algebraic varieties, *Siberian Advanced Mathematics, Allerton Press*, **7:2** (1997) 64–97.

[12] B. Plotkin, *Algebraic geometry in First Order Logic*, Sovremennaja Matematika and Applications **22** (2004), p. 16–62. Journal of Math. Sciences, **137**, n.5, (2006), p. 5049– 5097. http:// arxiv.org/ abs/ math GM/0312485.

[13] B. Plotkin, *Algebraic logic and logical geometry in arbitrary varieties of algebras*, In: Proceedings of the Conf. on Group Theory, Combinatorics and Computing, AMS Contemporary Math series, **611** (2014), 151–167.

[14] B. Plotkin, E. Aladova, E. Plotkin, *Algebraic logic and logically-geometric types in varieties of algebras.* Journal of Algebra and its Applications, **12(2)** Paper No. 1250146, 23 p. (2013).

[15] B. Plotkin, E. Plotkin, *Multi-sorted logic and logical geometry: some problems.* Demonstratio Mathematica, to appear.

[16] B. Plotkin, T. Plotkin, *Geometrical aspect of databases and knowledge bases.* Algebra Universalis **46** (2001), 131–161.

[17] B. Plotkin, T. Plotkin, *Categories of elementary sets over algebras and categories of elementary algebraic knowledge.* LNCS, Springer-Verlag, **4800** (2008), 555–570.

[18] J.D.H.Smith, An Introduction to Quasigroups and their Representations. Chapman and Hall/CRC Press. (2007).

[19] G. Zhitomirski, On logically-geometric types of algebras, preprint arXiv: 1202.5417v1 [math.LO].